Tensor Valuations in Stochastic Geometry and Imaging

Abstracts

Geometric inequalities in Hermitian vector spaces. Part II: Aleksandrov-Fenchel inequalities

Judit Abardia
Goethe-Universität Frankfurt

In this talk I will present the Aleksandrov-Fenchel type inequality satisfied by the Hermitian quermassintegrals introduced in Part I by T. Wannerer. The isoperimetric inequalities described before together with a Hermitian extension of the Brunn-Minkowksi inequality can be deduced from this new Aleksandrov-Fenchel type inequality. The main step of the proof of the Aleksandrov-Fenchel type inequality consists of associating a second order differential operator to each unitary-invariant valuation.

This is joint work with T. Wannerer.

Structures on valuations

Semyon Alesker
Tel Aviv University

In recent years several new structures have been discovered on the space of valuations. Besides their intrinsic importance for the valuation theory, they turned out to be useful in applications to integral geometry. In my talks I will outline constructions and properties of product and convolution on translation invariant valuations, and the Fourier type transform which relates between them. If time permits, I will also discuss exterior product, pull-back, and push-forward on translation invariant valuations. These operations refine product and convolution. Though I will focus on scalar valued valuations, understanding this case is necessary for further extensions to tensor valuations.

Concentration inequalities for random geometric graphs

Sascha Bachmann
Universität Osnabrück

We investigate a model of generalized random geometric graphs introduced by R. Lachièze-Rey and G. Peccati in their paper *Fine Gaussian fluctuations on the Poisson space, I (2013).* Consider a homogeneous Poisson point process η in $\mathbb{R}^d$, a compact window $W \subset \mathbb{R}^d$ and a symmetric set $S \subset \mathbb{R}^d$ of finite Lebesgue measure. The random geometric graph $G_S(\eta)$ has vertex set $\eta \cap W$ and an edge between two distinct vertices x and y whenever $x - y \in S$. For a given (finite and connected) graph H one can consider the corresponding *subgraph count*, i.e. the random variable that counts the occurrences of H as a subgraph of $G_S(\eta)$. New concentration inequalities for these random variables will be presented in this talk.

This is joint work with Matthias Reitzner.

Small cells in a random hyperplane mosaic

Gilles Bonnet
Universität Osnabrück

We will explain what is meant by the typical cell of a random stationary hyperplane mosaic in $\mathbb{R}^d$ and investigate its distribution. We will see how the distributions of the number of facets, the size and the shape of the typical cell are related. Finally we give some asymptotic results for the distribution of small cells.

Lecture 2: Topics in geometric inference

David Cohen-Steiner
INRIA, Sophia Antipolis

In geometric inference, we consider a compact subset of Euclidean space, and a discrete approximation of this set. The question is then to determine under which conditions is it possible to reliably recover geometric and topological information about the underlying object from the approximation only. This mini-course will consist in two largely independent lectures.

The second lecture will be about persistent homology, which was first introduced around 2000 by Edelsbrunner et. al., extending ideas previously known as size theory. This concept was later further extended in several ways and became the cornerstone of the nascent field of topological data analysis. In a nutshell, persistent homology essentially is an augmented version of Morse theory that enjoys good robustness properties. As for any mathematical tool meant to deal with data, its controlled behaviour under perturbations of the data is key to its relevance. These properties make it possible to infer various geometric properties of objects known only through an approximation, *e.g.* a set of possibly noisy sample points. The lecture will first cover the basic tools of algebraic topology that persistent homology builds upon, that is, homology groups. We will then move to the definition of persistent homology and state its main properties. Finally we will give applications to geometric inference, especially of Betti numbers and intrinsic volumes. If time permits, we will also mention recent results in the area.

Multiplication by kinematic valuations defined via WDC sets

Joseph Fu
University of Georgia

A *kinematic valuation* is a smooth valuation ν on a manifold M that may be expressed in terms of intersections with the images of a nice object $X \subset M$ under a sufficiently rich measured smooth family of diffeomorphisms of M. Using only advanced calculus we have recently shown that, if X is a smooth polyhedron, then the Alesker-Bernig formula for the product of ν with an arbitrary smooth valuation μ arises in a simple way from a classical decomposition of the normal cycle of the intersection of two smooth polyhedra that intersect transversely. In this talk we describe how this analysis extends to the more general setting in which X is a WDC set. Although the proof in this case relies on the more sophisticated language of the Federer-Fleming theory of currents, it is in other respects more straightforward. Specializing to an isotropic space (M, G), and replacing the general family of diffeomorphisms above by the action of G, we obtain the array of kinematic formulas for pairs of WDC sets in M.

Lecture 1: Local tensor valuations

Daniel Hug
Karlsruhe Institute of Technology

In the first talk, we will consider local tensor valuations for general convex bodies, continuing the discussion from Rolf Schneider's lecture. As explained there, for local tensor valuations on polytopes of fixed rank, an explicit basis is known. It is natural to expect that this can be extended to a basis of the vector space of all local tensor valuations of a fixed rank on general convex bodies. Surprisingly, this is not true in general. Instead, we present a complete characterization theorem and explain the phenomenon of "non-extendability" in this context.

Lecture 2: Integral geometric formulas for tensor valuations

Daniel Hug
Karlsruhe Institute of Technology

Integral geometric formulas for tensor valuations naturally extend formulas known from the scalar case, but in most cases the analysis is more involved and requires new methods and additional efforts. A first major step was the proof of general Crofton formulas for arbitrary tensor valuations by Hug-Schneider-Schuster. We will explain these results as well as the inherent problem related to these. In the case of translation invariant valuations, the picture could be clarified recently. Combining methods of algebraic integral geometry, differential geometry and previously known results, general integral geometric formulas of Crofton, additive and kinematic type could be established in this setting (Bernig-Hug). If time permits, we will also comment on simplified "intrinsic" Crofton formulas for translation invariant tensor valuations (Weis-Hug) which should be particularly relevant for applications in stereology.

The method of densities for non-isotropic Boolean models

Julia Hörrmann
Karlsruhe Institute of Technology

The Boolean model is the basic random set model for the description of porous structures like the pore space in sandstone or bread or for the solid phase of sintered ceramic composites. It is obtained by decorating the points of a homogeneous Poisson point process (the germs) with independent identically distributed random compact, convex particles (the grains) and forming the union set. If a Boolean model is fitted to a real structure the first task is to estimate the intensity from observable quantities.

In the isotropic case the Miles formulas can be used to express the intensity in terms of the densities of the intrinsic volumes (in two dimensions these are the volume, the half of the surface area and the Euler characteristic). This approach is known as the method of densities.

In the more difficult non-isotropic situation Weil showed in two and three dimensions that the intensity is uniquely determined by the densities of the mixed volumes. Combining Weil's ideas of translative integral geometry with ideas from harmonic analysis and approximation theory we obtain formulas for the intensity which are directly comparable to the Miles formulas. For this purpose we introduce a new collection of geometric functionals on the space of convex bodies, the *harmonic intrinsic volumes*. Under a regularity assumption on the grain distribution we obtain a series representation of the intensity in terms of the densities of the harmonic intrinsic volumes.

Moreover, we introduce a *modulus of isotropy* of the grain distribution to quantify the degree of anisotropy of the Boolean model. If the intensity is approximated by the truncated series depending on the densities of only finitely many harmonic intrinsic volumes, we can bound the error term from above using the modulus of isotropy.

J. Hörrmann. The method of densities for non-isotropic Boolean models. (in preparation).

Characterization of maximally random jammed sphere packings using Voronoi correlation functions

Michael Klatt
Universität Erlangen-Nürnberg

We characterize the structure of maximally random jammed (MRJ) sphere packings by computing the Minkowski functionals of their associated Voronoi cells. However, they incorporate only local information and are insensitive to global structural information. Therefore, we extend this by evaluating descriptors that incorporate non-local information like correlation functions of the Minkowski functionals. We ascertain these higher-order functions on our MRJ packings as well as hard spheres in equilibrium and the Poisson point process.

Surface tensor estimation from linear sections

Astrid Kousholt
Aarhus University

From Crofton's formula for Minkowski tensors we derive stereological estimators of translation invariant surface tensors of convex bodies in the n-dimensional Euclidean space. The estimators are based on measurements in one-dimensional linear sections of the underlying convex body. In a design based setting we discuss three types of estimators. The first type of estimators are based on isotropic uniform random lines. Due to the structure of the measurement function it suffices to observe whether the test line hits or misses the underlying convex body in order to estimate the surface tensors. However, the resulting estimators posses some unfortunate statistical properties, and we will discuss methods of improvement. The second type of estimators are based on non-isotropic random lines. These estimators are developed with inspiration from a well-known fact from the theory of importance sampling, namely that variance reduction of estimators can be obtained by modifying the sampling distribution in a suitable way. If time allows, we will further discuss estimators based on vertical sections.

This is joint work with Markus Kiderlen and Daniel Hug.

Valuations on lattice polytopes

Monika Ludwig
Technische Universität Wien

Lattice polytopes are convex hulls of finitely many points with integer coordinates in $\mathbb{R}^n$. The classification of real-valued invariant valuations on lattice polytopes by Betke & Kneser is classical (and will be discussed in the first part of these talks). Building on this, a complete classification is established of Minkowski valuations on lattice polytopes that intertwine the special linear group over the integers and are translation invariant. In the contravariant case, the only such valuations are multiples of projection bodies. In the equivariant case, the only such valuations are generalized difference bodies combined with multiples of the newly defined discrete Steiner point. More general invariant tensor valuations are defined.

This is joint work with Károly J. Böröczky.

Lecture 1: Topics in geometric inference

Quentin Mérigot
CNRS, Université Paris Dauphine

In geometric inference, we consider a compact subset of Euclidean space, and a discrete approximation of this set. The question is then to determine under which conditions is it possible to reliably recover geometric and topological information about the underlying object from the approximation only. This mini-course will consist in two largely independent lectures.

In the first lecture, we will focus on the inference of normals, curvature measures, and location of sharp edges. In a first part, we will review the tube formula, Federer's curvature measures, and a stability result for these curvature measures relying on convex analysis. In a second part, we will introduce the Voronoi covariance measure of a compact set of $\mathbb{R}^d$, which is a tensor-valued measure. The VCM is also Hausdorff-stable, and can be used for the inference of differential quantities such as normals and the location of sharp edges. Finally, in a third part, we will introduce the notion of distance to a measure, which shares many properties with the distance function to a compact set but is more robust to noise and outliers. We will explain how to extend the VCM construction to this section and present some computational results.

Continued fractions built from convex sets and convex functions

Ilya Molchanov
University of Bern

The talk is devoted to convergence results for continued fractions built from a sequence of convex sets by using the polarity transform instead of the one-over transformation and the Minkowski addition as the sum. In a similar manner, continued fractions are built from a sequence of convex functions using the Legendre-Fenchel or Artstein-Avidan-Milman transforms. The both cases can be treated as special variants of general results for continued fractions on semigroups also introduced in this talk. The case of random terms is also covered – it yields new distributions of random convex bodies.

Moments and valuations

Lukas Parapatits
ETH Zürich

The moment vector is a fundamental quantity of a convex body. It was previously characterized by Monika Ludwig as essentially the only measurable $\mathrm{GL}(n)$ covariant vector-valued valuation on convex polytopes containing the origin in their interiors. In a joint work with Christoph Haberl, we were able to replace the assumption of $\mathrm{GL}(n)$ covariance with $\mathrm{SL}(n)$ covariance. This is the first step in a process that will hopefully result in an analogous characterization of certain tensor-valued valuations in the future.

Construction and application of uniformly distributed sequences in the orthogonal group

Florian Pausinger
IST Austria

Quasi-Monte Carlo methods replaced classical Monte Carlo methods in many areas of numerical analysis over the last decades. The purpose of this talk is to extend quasi-Monte Carlo methods into a new direction. Motivated by the question of how to numerically approximate Crofton-type integrals, we present recent progress concerning the explicit construction and implementation of uniformly distributed sequences in the orthogonal group and on the

Grassmannian manifold. We show that our deterministic sequences compare well with classical random constructions, thus motivating various directions for future research.

Kinematic formulae for d.c. domains

Dušan Pokorný
Charles University, Prague

We show that the curvature measures of any two WDC sets (sublevel sets of d.c. functions at weakly regular values) fulfil the kinematic formulae of integral geometry. An important step in the proof is to show that a properly defined unit normal bundle of a WDC set in $\mathbb{R}^d$ has σ-finite $(d-1)$-dimensional Minkowski content. This follows using a duality argument from convex geometry from a modified version of a result of Evald, Larman and Rogers about directions of segments lying on the boundary of a convex body.

This is joint work with Joseph Fu and Jan Rataj.

Curvature measures of d.c. domains

Jan Rataj
Charles University, Prague

A d.c. function is a difference of two convex functions, and a d.c. domain in $\mathbb{R}^d$ is a subset which can be locally described as subgraph of a d.c. function. We show that a d.c. domain admits curvature measures that preserve the usual properties as additivity, local determinancy, Gauss-Bonnet formula and Crofton formula. In fact, the procedure works with a larger class of so called WDC sets encompassing e.g. sets with positive reach. The construction is based on approximation with smooth functions.

This is joint work with Dušan Pokorný.

Lectures 1+2: Valuations on convex bodies – the classical basic facts

Rolf Schneider
Albert-Ludwigs-Universität Freiburg

This is an introduction to valuations on convex bodies and a survey over the ‘state of the art’, as it was before the modern development, which began in the second half of the 1990's. We take two historical origins as starting points for presenting, first, the theory of valuations on polytopes, and second, the fundamental facts on translation invariant, continuous valuations on convex bodies. Extension theorems, the decomposition theorem of McMullen, and classification theorems for continuous valuations under additional assumptions are the main results. These include Hadwiger's characterization theorem for the intrinsic volumes.

As a natural generalization of the intrinsic volumes, we then introduce the Minkowski tensors, which are no longer translation invariant, but show a simple polynomial behaviour under translations.

Lecture 3: Tensor valuations

Rolf Schneider
Albert-Ludwigs-Universität Freiburg

Continuing the treatment of Minkowski tensors, we present the McMullen relations and describe Alesker's characterization theorem. Then we pass over to local tensor valuations and explain a classification theorem for these, under suitable assumptions, on the class of polytopes.

Characterizing anisotropic features of cellular and granular structures by Minkowski tensors

Gerd Schröder-Turk
Universität Erlangen-Nürnberg

For studies of physical systems characterized by the presence of a disordered spatial structure, the question "How do you quantitatively measure the shape of a random structure?" is a question of great importance. More specific questions of a similar nature are "How do you measure the differences between two similar random structures?" or "What is a good (i.e. succinct) measure to evaluate changes of an evolving random spatial structure?". In this talk, we argue that Minkowski functionals and Minkowski tensors [1-3] provide a useful method to address these questions. For the sake of specificity, we focus on two specific systems namely granular materials (aka sphere packs, or hard-core particle processes, [4-6]) and random liquid foams (aka random tessellations [7,8]). For these systems, we provide numerical data that demonstrate the ability of the Minkowski tensor approach to provide at least partial answers to the two latter questions above.

This lecture is based on joint work with Fabian Schaller, Sebastian Kapfer, Michael Klatt, Walter Mickel, and Klaus Mecke.

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Second order properties and central limit theorems for geometric functionals of Boolean models

Matthias Schulte
Karlsruhe Institute of Technology

Boolean models are a fundamental topic of stochastic geometry and continuum percolation and have applications, for example, in physics, materials science and biology. The stationary Boolean model Z is the random closed set that is formed by the union of compact and convex particles generated by a stationary Poisson process. The goal of this talk is to investigate random variables of the form $\psi(Z \cap W)$, where ψ is an additive, translation invariant and locally bounded functional and W is a compact and convex observation window. Examples for ψ are volume, surface area and Euler characteristic. For increasing observation windows asymptotic covariances are computed and univariate and multivariate central limit theorems are derived. In the important special case of intrinsic volumes of an isotropic Boolean model the covariance formulas can be further simplified. The proofs combine the Fock space representation and the Malliavin-Stein method with some results from convex and integral geometry.

This is joint work with D. Hug and G. Last.

Harmonic analysis of translation invariant valuations

Franz Schuster
Technische Universität Wien

Hadwiger's characterization of continuous rigid motion invariant real valued valuations has been the starting point for many important recent developments in valuation theory. In this mini-course, I will present the decomposition of the space of continuous and translation invariant valuations into a sum of $\mathrm{SO}(n)$ irreducible subspaces. I will explain how this result can be reformulated in terms of a Hadwiger type theorem for translation invariant and $\mathrm{SO}(n)$ equivariant valuations with values in any finite dimensional $\mathrm{SO}(n)$ module. In particular, this includes valuations with values in tensor spaces. If time permits, I will also present applications to the theory of bivaluations.

This is joint work with Semyon Alesker and Andreas Bernig.

Voronoi-based estimation of Minkowski tensors from digital images

Anne Marie Svane
Aarhus University

Minkowski tensors are used in shape analysis to quantify such properties as position, orientation and eccentricity of an object. Other important special cases are the intrinsic volumes, including surface area, integrated mean curvaure, and Euler characteristic. Often, the only information about the object is a digital image, which makes exact computations impossible. Moreover, most known estimation procedures are either biased or very complex. In this talk we shall consider a class of digital algorithms for estimation of all Minkowski tensors based on the Voronoi decomposition associated with the image. These algorithms can be shown to converge when the resolution goes to infinity.

This is joint work with Markus Kiderlen and Daniel Hug.

Geometric inequalities in Hermitian vector spaces. Part I: Isoperimetric inequalities

Thomas Wannerer
Goethe-Universität Frankfurt

In this talk I will present a recent generalization of the classical isoperimetric inequalities for quermassintegrals in hermitian vector spaces. Consider the Grassmannian of real k-planes in $\mathbb{C}^n$. For $k = 2, 3$ and $n \geq k$ the action of the unitary group decomposes the Grassmannian into infinitely many orbits parametrized by a single real parameter, known as the Kähler angle $\theta \in [0, \pi/2]$. Each orbit gives rise to a quermassintegral by averaging the volume of projections onto the planes of the orbit. For $\theta \in [\pi/4, \pi/2]$ and hence in particular for orbits of isotropic or Lagrangian planes, the corresponding quermassintegrals satisfy isoperimetric inequalities similar to the ones discovered by Minkowksi, Aleksandrov, and Fenchel.

This is a joint work with Judit Abardia.

Kinematic formulas for area measures

Wolfgang Weil
Karlsruhe Institute of Technology

The Principal Kinematic Formula and the Crofton Formula are classical results for the intrinsic volumes of convex bodies. Both formulas hold true in a local version, for curvature measures. Here, we show that there are also local variants for area measures. These kinematic formulas involve certain Fourier operators which also arise in the study of mean section bodies.

The talk is based on joint work with Paul Goodey and Daniel Hug.

Scaling exponents of curvature measures

Steffen Winter
Karlsruhe Institute of Technology

I will present some recent joint work with Dušan Pokorný. We study the scaling behaviour of the curvature measures of the parallel sets F_r of a compact set F in $\mathbb{R}^d$ as the parallel radius r tends to zero. We are particularly interested in the situation when F is a self-similar set and the curvature measures $C_k(F_r, \cdot)$ do not scale in the generic way – namely like r^{k-D}, where D is the Minkowski dimension of F. We give a characterization of nongeneric behaviour in $\mathbb{R}$ and $\mathbb{R}^2$ based on the notion of local flatness and also some results in higher dimension. The study revealed some fundamental open questions concerning the curvature measures of parallel sets in general, e.g. the dependence of the scaling behaviour on the dimension of the ambient space. Some of the problems are solved in special cases, e.g. in $\mathbb{R}^2$, while the general case remains open.