Kinematic Formulas for Area Measures

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Sandbjerg September 2014

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1. Classical kinematic formulas

The classical kinematic integral formulas in Integral Geometry are the **Principal Kinematic Formula (PKF)** and the **Crofton Formula (CF)** for intrinsic volumes V_j of convex bodies (or polyconvex sets) $K, M \subset \mathbb{R}^d$:

$$\int_{G_d} V_j(K \cap gM)\, \mu(dg) = \sum_{k=j}^{d} \tilde{c}(d,j,k) V_k(K) V_{d+j-k}(M),$$

$j = 0, \ldots, d$, and

$$\int_{A(d,q)} V_j(K \cap E)\, \mu_q(dE) = \tilde{c}(d,j,q) V_{d+j-q}(K).$$

$j = 0, \ldots, q$.

In their local variant for **curvature measures**, we have

$$\int_{G_d} C_j(K \cap gM, A \cap gB)\, \mu(dg) = \sum_{k=j}^{d} \tilde{c}(d,j,k) C_k(K,A) C_{d+j-k}(M,B),$$

$j = 0, \ldots, d$, and

$$\int_{A(d,q)} C_j(K \cap E, A \cap E)\, \mu_q(dE) = \tilde{c}(d,j,q) C_{d+j-q}(K,A),$$

$j = 0, \ldots, q$, for convex bodies K, M and Borel sets $A, B \subset \mathbb{R}^d$.

How about kinematic formulas for the other local analogs of the intrinsic volumes, the **area measures**?

2. Area measures

A short reminder to the definition of area measures:

Let $K \subset \mathbb{R}^d$ be a **convex body** (a non-empty compact convex set).

The **surface area measure** $S_{d-1}(K, \cdot)$ of K is a measure on the unit sphere S^{d-1}. For a Borel set $A \subset S^{d-1}$, $S_{d-1}(K, A)$ measures the area (Hausdorff measure) of the set of boundary points of K, which have an outer unit normal in A.

By the **local Steiner formula**,

$$S_{d-1}(K + rB^d, \cdot) = \sum_{j=0}^{d-1} r^{d-1-j} \binom{d-1-j}{j} S_j(K, \cdot)$$

the **lower order area measures** $S_0(K, \cdot)$(= spherical Lebesgue measure σ) and $S_1(K, \cdot), ..., S_{d-2}(K, \cdot)$ are introduced.

It is easy to see that kinematic formulas for area measures cannot hold in exactly the same form, hence they have to be modified appropriately.

E.g., a CF of the form

$$\int_{A(d,q)} S_j(K \cap E, A \cap L(E))\, \mu_q(dE) = \tilde{c}(d,j,q) S_{d+j-q}(K,A),$$

is wrong if K is a polytope and A is the support of $S_{d+j-q}(K,\cdot)$, since the integrand then vanishes for almost all E.

The same is true, if $S_j(K \cap E, \cdot)$ is replaced by the intrinsic version $S_j'(K \cap E, \cdot)$.

Hence, we consider

$$\int_{A(d,q)} S_j(K \cap E, \cdot)\, \mu_q(dE),$$

for $1 \leq j < q \leq d-1$, and ask whether it can be expressed by an area measure of K?

Two related results are due to **Glasauer (1997)**, who showed that

$$\int_{G_d} \Theta_j(K \cap gM, A \wedge gB))\, \mu(dg) = \sum_{k=j}^{d} \tilde{c}(d,j,k) \Theta_k(K,A) \Theta_{d+j-k}(M,B),$$

for $j = 0, \ldots, d$, and

$$\int_{A(d,q)} \Theta_j(K \cap E, A \wedge E)\, \mu_q(dE) = \tilde{c}(d,j,q) \Theta_{d+j-q}(K,A),$$

for $j = 1, \ldots, q-1$. Here $\Theta_i(K, \cdot)$ is the i-th **support measure** of K, A, B are Borel sets in the normal bundle of K, M and $\wedge$ is a suitable law of composition.

Although the area measures are projections of the support measures, these results do not imply explicit formulas of kinematic type for the area measures (due to the definition of $\wedge$).

3. Fourier operators

In **Goodey-Yaskin-Yaskina (2009)**, the authors studied operators I_p, for $0 < p < d$, defined on functions $f \in C^\infty(S^{d-1})$ as follows:

Let f_p be the homogeneous (of degree $-d+p$) extension of f to $\mathbb{R}^d \setminus \{o\}$ and let $\widehat{f}_p$ be its distributional Fourier transform. The restriction of $\widehat{f}_p$ to S^{d-1} is again a smooth (complex) function and we consider the operator

$$I_p : f \mapsto \widehat{f}_p|_{S^{d-1}}.$$

Since I_p intertwines the group action of $SO(d)$, it acts as a multiple of the identity on the spaces H_n^d of spherical harmonics (of degree $n = 0, 1, \ldots$). The multipliers are given by

$$\lambda_n(d,p) = \pi^{d/2} 2^p (-1)^{n/2} \frac{\Gamma(\frac{n+p}{2})}{\Gamma(\frac{n+d-p}{2})}. \quad (1)$$

For even n they are real, for odd n they are purely imaginary.

Due to (1), the operator I_p can be defined (by analytic continuation) for various other (complex) values of p, in particular for $p = -1$, but then only on functions without linear spherical harmonic (**centred functions**).

For the composition $I_q I_p$, all multipliers are real and non-zero (if $q = -1$ only for $n \neq 1$). Thus we have a bijection

$$I_q I_p : C^\infty(S^{d-1}) \to C^\infty(S^{d-1})$$

(for $p, q \neq -1$), respectively

$$I_{-1} I_p : C_0^\infty(S^{d-1}) \to C_0^\infty(S^{d-1})$$

(where $C_0^\infty(S^{d-1})$ is the space of centred functions in $C^\infty(S^{d-1})$).

Also, for $1 \leq p \leq d$, we have

$$I_{d-p} I_p = (2\pi)^d I^*,$$

where $(I^* f)(u) = f(-u)$.

Since I_p is self-adjoint, it extends to mapping on distributions. Here,

$$\Box I_{-1} = -\frac{1}{d-1} I_1,$$

where $\Box$ is the differential operator which satisfies the distributional equation $\Box h(K, \cdot) = S_1(K, \cdot)$ for all convex bodies K.

4. Mean section bodies

For $1 \leq k \leq d-1$, the k-th **mean section body** $M_k(K)$ of a **convex body** K in $\mathbb{R}^d$ was introduced in **Goodey-W. (1992)** as the Minkowski sum of all sections of K by k-dimensional (affine) flats. In terms of **support functions**,

$$h(M_k(K), \cdot) = \int_{A(d,k)} h(K \cap E, \cdot) \mu_k(dE),$$

where $A(d,k)$ is the affine Grassmannian and μ_k is the motion invariant measure on $A(d,k)$.

For simplicity, we assume $\dim K = d$ and $k \geq 2$ ($M_1(K)$ is always a ball). Also, we may center the support functions, h^*, by requiring that the bodies have their Steiner point at the origin.

For $k = 2$, we showed

$$h^*(M_2(K), u) = c_d \int_{S^{d-1}} \alpha(x,u) \sin \alpha(x,u)\, S_{d-1}(-K, dx),$$

where $\alpha(x,u)$ is the (smaller) angle between x and u. This is generalized by the following result from **Goodey-W. (2014)**:

Theorem. *For $k = 2, \ldots, d$, we have*

$$h^*(M_k(K), \cdot) = c_{d,k} I_{-1} I_{k-1} S_{d+1-k}(-K, \cdot).$$

Corollary 1. $M_k(K)$ determines K uniquely.

Corollary 2. We have

$$S_{d+1-k}(K, \cdot) = \bar{c}_{d,k} I_{d-1} I_{d+1-k} S_1(-M_k(K), \cdot).$$

The **proof** of the theorem is quite involved and works by induction using the I_p-operators in lower dimensional spaces together with results on spherical projections and liftings from **Goodey-Kiderlen-W. (2011)**. Also, it is based on an exchange formula for mean section bodies which follows from a result of **Alesker-Bernig-Schuster (2011)**.

5. Kinematic formulas for area measures

Concerning a Crofton-type result, we show:

Theorem. *For convex bodies K and Borel sets $A \subset S^{d-1}$, we have*

$$\int_{A(d,q)} S_j(K \cap E, \cdot)\, \mu_q(dE) = a_{d,j,q} I_j I_{q-j} S_{d+j-q}(-K, \cdot),$$

for $1 \leq j < q \leq d-1$ with explicitly given constants.

Outline of the proof:

From the second corollary (applied to $K \cap E$) and using the linearity of the first area measure, we get (with a changing constant c)

$$\begin{aligned}
&\int_{A(d,q)} S_j(-(K \cap E), \cdot)\, \mu_q(dE) \\
&\qquad = c \int_{A(d,q)} I_{d-1} I_j S_1(M_{d+1-j}(K \cap E), \cdot)\, \mu_q(dE) \\
&\qquad = c I_{d-1} I_j \int_{A(d,q)} \int_{A(d,d+1-j)} S_1(K \cap E \cap F), \cdot) \\
&\qquad\quad \times\ \mu_{d+1-j}(dF)\, \mu_q(dE) \\
&\qquad = c I_{d-1} I_j \int_{A(d,q+1-j)} S_1(K \cap H, \cdot)\, \psi(dH),
\end{aligned}$$

where ψ is the image measure (on $A(d, q+1-j)$) of $\mu_{d+1-j} \otimes \mu_q$ under the (almost everywhere defined) mapping $(E, F) \mapsto E \cap F$.

We have $g(E \cap F) = gE \cap gF$ and so ψ is motion invariant and hence a multiple of μ_{q+1-j}. Therefore,

$$\begin{aligned}
&\int_{A(d,q)} S_j(-(K \cap E), \cdot)\, \mu_q(dE) \\
&\qquad = c I_{d-1} I_j \int_{A(d,q+1-j)} S_1(K \cap H, \cdot)\, \mu_{q+1-j}(dH) \\
&\qquad = c I_{d-1} I_j S_1(M_{q+1-j}(K), \cdot) \\
&\qquad = c I_j I_{q-j} S_{d+j-q}(K, du),
\end{aligned}$$

where we have used the linearity of the first area measure and the second corollary again (as well as the inversion formula for I_p).

In order to obtain a corresponding kinematic formula, we use a local version of Hadwiger's general integral geometric theorem:

Theorem. *Let $\varphi : \mathcal{K}' \to \mathcal{M}^+(S^{d-1})$ be a continuous and additive mapping. Then, for $K, M \in \mathcal{K}'$ and Borel sets $A \subset S^{d-1}$,*

$$\int_{G_d} \varphi(K \cap gM, A)\, \mu(dg) = \sum_{k=0}^{d} [T_{d,k}\varphi(K, \cdot)](A) V_k(M),$$

with mappings $T_{d,k} : \mathcal{M}^+(S^{d-1}) \to \mathcal{M}^+(S^{d-1})$ which are given by the Crofton integrals

$$T_{d,k}\varphi(K, \cdot) = \int_{A(d,k)} \varphi(K \cap E, \cdot)\, \mu_k(dE), \quad k = 0, \ldots, d.$$

From the CF, we thus get a PKF for area measures:

$$\int_{G_d} S_j(K \cap gM, A)\, \mu(dg) = \sum_{k=j}^{d} a_{d,j,k}[I_j I_{k-j} S_{d+j-k}(-K, \cdot)](A) V_k(M),$$

for $1 \leq j \leq d-1$.

References (in chronological order)

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